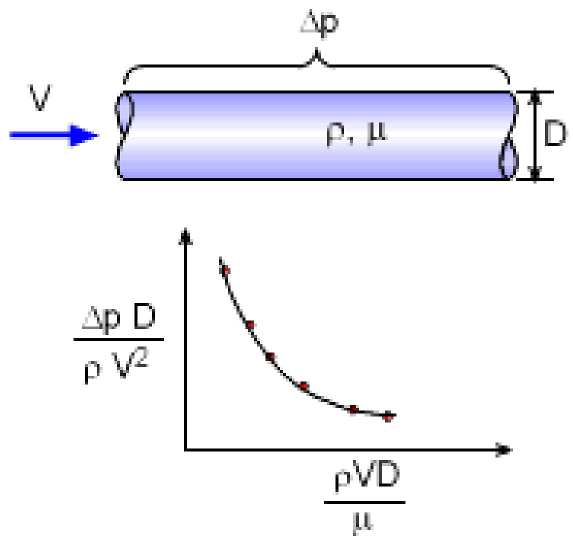
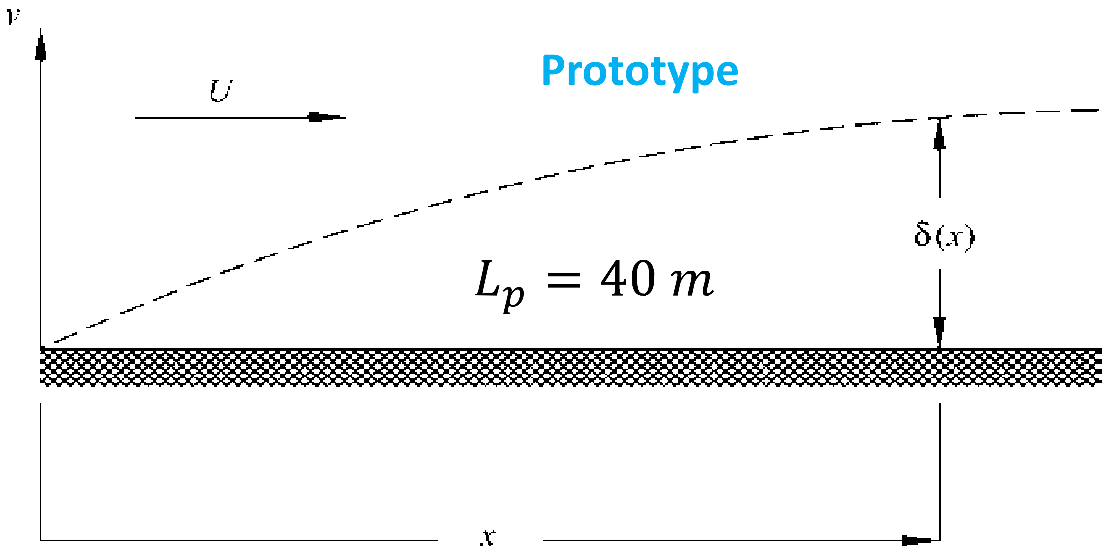
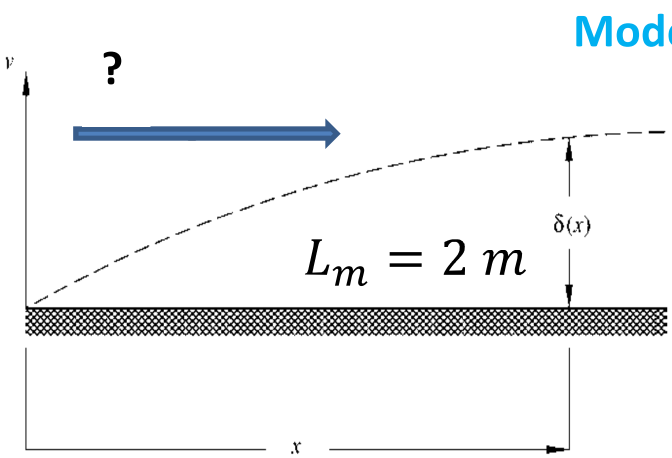
Dimensional analysis and Similarity

Problems and solutions



Drag force Similarity → Laboratory experiment





Similarity

$$C_D = f(R_L)$$

Reynold similarity; $R_L|_p = R_L|_m = R_L$ ⟹ $C_D\big|_p = C_D\big|_m$

Same fluid $\mu_m = \mu_p = \mu, \; and \; \rho_m = \rho_p = \rho$

$$R_L\Big|_p = \frac{\rho U_p L_p}{\mu}$$

$$R_L\Big|_m = \frac{\rho U_m L_m}{\mu}$$

⟹ $$U_m = U_p \frac{L_p}{L_m}$$

Experimental value (D_m)

$$C_D\Big|_m = \frac{D_m}{\rho\, b_m L_m U_m^2/2} = C_D\Big|_p$$

Prototype drag ⟹ $D_p = \frac{1}{2}\rho\, b_p L_p U_p^2 \; C_D\Big|_m$; since

Different fluids $\mu_m \neq \mu_p,\ and\ \ \rho_m \neq \rho_p$

$$R_L|_p = R_L|_m = R_L \quad \Rightarrow \quad C_D\big|_p = C_D\big|_m$$

$$R_L\big|_p = \frac{\rho_p U_p L_p}{\mu_p}$$

$$R_L\big|_m = \frac{\rho_m U_m L_m}{\mu_m}$$

$$\Rightarrow \quad U_m = U_p \frac{\rho_p L_p \mu_m}{\rho_m L_m \mu_p}$$

Experimental value

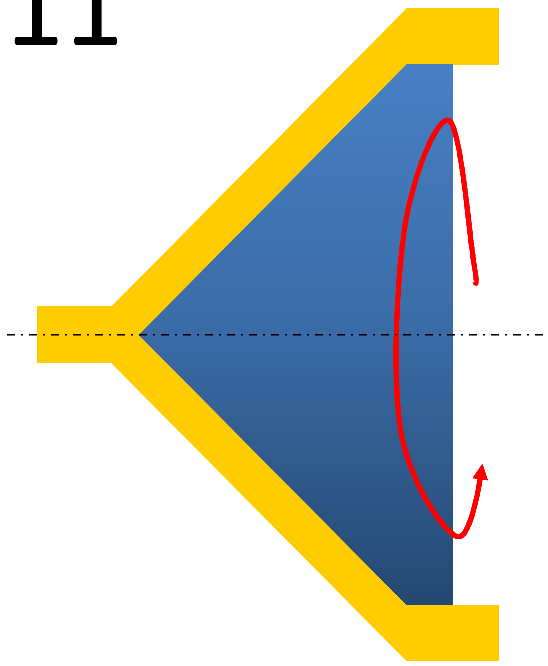
$$C_D\big|_m = \frac{D_m}{\rho_m\, b_m L_m U_m^2/2} = C_D\big|_p$$

Prototype drag $\Rightarrow$ $D_p = \frac{1}{2}\rho_p b_p L_p U_p^2\ C_D\big|_m$

Worked example 11

If the rotating cone has a base diameter of 120mm and is to be rotated first in ambient air and subsequently in water. What rotational speed should be used in water to obtain the same Reynolds number as that obtained in air at 3000 rpm? Take the density and viscosity of air to be 1.2 kg/m^3 and $1.8x10^{-5}$ kg/ms respectively and the density and viscosity of water to be 1000 kg/m^3 and 0.001 kg/ms respectively.

Answer: 200 rpm



Solution to Worked Example 11

Reynolds similarity $$Re = \frac{\rho \omega D^2}{\mu}$$

$$Re_{air} = \frac{1.2 \times \left(3000 \times \frac{2\pi}{60}\right) \times 0.12^2}{1.8 \times 10^{-5}} = 3.02 \times 10^5$$

$$Re_{water} = \frac{1000 \times \left(N \times \frac{2\pi}{60}\right) \times 0.12^2}{0.001}$$

$$= 3.02 \times 10^5$$

→ N = 200 rpm

Solution to Worked Example 11

Relation between air and water rotational speed

$$Re_{air} = \left(\frac{\rho\omega D^2}{\mu}\right)_{air} = Re_{water} = \left(\frac{\rho\omega D^2}{\mu}\right)_{water}$$

$$Re_{air} = \left(\frac{\rho N \left\{\frac{2\pi}{60}\right\} D^2}{\mu}\right)_{air} = Re_{water} = \left(\frac{\rho N \left\{\frac{2\pi}{60}\right\} D^2}{\mu}\right)_{water} \Rightarrow \left(\frac{\rho N}{\mu}\right)_{air} = \left(\frac{\rho N}{\mu}\right)_{water}$$

$$N_{water} = N_{air} \times \frac{\rho_{air}}{\rho_{water}} \frac{\mu_{water}}{\mu_{air}} = \frac{N_{air}}{15}$$

Use to build our model

In the present example

N_{air} **= 3000 rpm then** N_{water} **= N = 200 rpm**

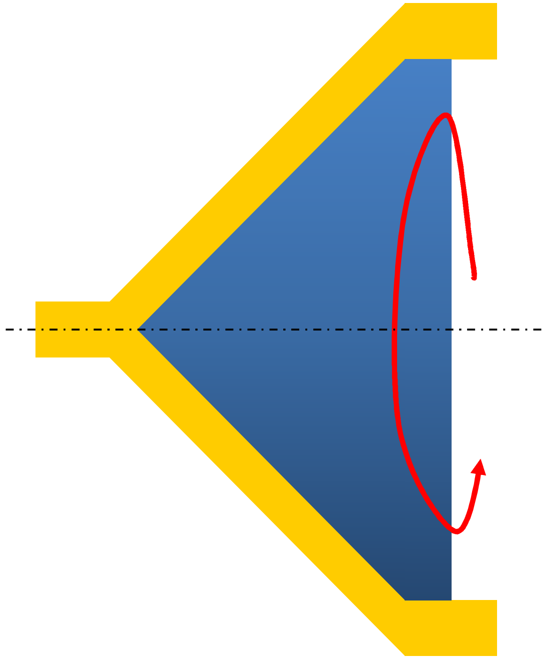
Worked example 12

The torque on the rotating cone shown is related to the rotation speed, fluid properties and cone base diameter:

$$T = f(\omega, \rho, \mu, D)$$

Identify the appropriate non-dimensional groups.

Answer: Cm, Re



Solution to Worked Example 12

$$T = f(\omega, \rho, \mu, D)$$

- First find the dimensions of each variable

Quantity	SI unit	Dimensions
ρ, density	kg/m^3	ML^{-3}
ω, angular velocity	rad/s	T^{-1}
D, length or diameter	m	L
μ, viscosity	kg/ms	$ML^{-1}T^{-1}$
T, torque	Nm	ML^2T^{-2}

- 5 variables (n = 5)
- 3 dimensions (k = 3)
- M=5-3=2 dimensionless groups
- I choose my two non repeating variables as T and μ
- The repeating variables used to form the Π-groups are ρ, D and ω. These cannot be used to form a non-dimensional group on their own and so are suitable.

Choose T because it is the output variable

$$Re = \frac{\rho\omega D^2}{\mu}$$

Solution to Worked Example 12

- The two Π groups are

$$[\Pi_1] = [T][\rho]^a[D]^b[\omega]^c = [M^0L^0T^0]$$

$$[\Pi_1] = [ML^2T^{-2}][ML^{-3}]^a[L]^b[T^{-1}]^c = [M^0L^0T^0]$$

- Remember maths from way back – when you multiply you add the exponents?

$$x^a x^b = x^{a+b}$$

- We can now look at each of the dimensions in turn and balance the exponents
- For example taking [M]

- 1 + a = 0 so a = -1

$$M^1 M^a = M^0$$

Solution to Worked Example 12

- Similarly for L and T
- L: $2 - 3a + b = 0$
- T: $-2 - c = 0$ so c=-2
- Thus b = -5 (from the L equation)
- We have $[\Pi_1] = [\mathrm{T}][\rho]^{a}[\mathrm{D}]^{b}[\omega]^{c}$
- So $\Pi_1 = \mathrm{T}\rho^{-1}\mathrm{D}^{-5}\omega^{-2} = \dfrac{T}{\rho\omega^2 D^5}$
- Compare this to the torque coefficient $Cm = \dfrac{T}{\frac{1}{2}\rho\omega^2 D^5}$
- Buckingham-Π cannot identify constants.
- Our first non-dimensional group is Cm $\boldsymbol{C_m = \dfrac{T}{\frac{1}{2}\rho(\omega D)^2 D^2 D}}$

Solution to Worked Example 12

- We can follow the same process for the second non-dimensional group:

$$[\Pi_2] = [\mu][\rho]^d[D]^e[w]^f = [M^0L^0T^0]$$

$$[\Pi_2] = [ML^{-1}T^{-1}][ML^{-3}]^d[L]^e[T^{-1}]^f = [M^0L^0T^0]$$

- Again balancing the exponents:
- M: 1+d=0 so d=-1
- L: -1-3d+e=0
- T: -1-f=0 so f=-1
- Thus e=-2

$$\Pi_2 = \mu\rho^{-1}D^{-2}\omega^{-1} = \frac{\mu}{\rho\omega D^2}$$

As we saw before *Re* is one of the dominant similar parameters in the governing equations and need to be included in the analysis

- Compare this to the rotating Reynolds number
- Π_2 is 1/Re (which is also dimensionless of course).

$$Re = \frac{\rho\omega D^2}{\mu}$$

The functional relationship is therefore *Cm=f(Re)*

Problem:

On worked example 12 we are looking for the measurement of the **pressure distribution** inside the cone, where the pressure on the rotating cone is related to the rotation speed, fluid properties, viscosity μ and density ρ, and cone base diameter:

$$P = f(\omega, \rho, \mu, D)$$

Quantity	SI unit	Dimensions
ρ, density	kg/m^3	ML^{-3}
ω, angular velocity	rad/s	T^{-1}
D, length or diameter	m	L
μ, viscosity	kg/ms	$ML^{-1}T^{-1}$
P, pressure	N/m^2	$ML^{-1}T^{-2}$

- 5 variables (n = 5)
- 3 dimensions (k = 3)
- M=5-3=2 dimensionless groups
- I choose my two non repeating variables as P (objective function) and μ
- The repeating variables used to form the Π-groups are ρ, D and ω.

$$\pi_1 = P\,\rho^{a1}\omega^{a2}D^{a3}$$

$$\pi_1:\ [ML^{-1}T^{-2}][ML^{-3}]^{a1}[T^{-1}]^{a2}[L]^{a3} = [M^0L^0T^0]$$

$$M:\quad 1 + a1 = 0; \quad a1 = -1$$

$$L:\quad -1 - 3a1 + a3 = 0; \quad 2 + a3 = 0; \quad a3 = -2$$

$$T:\quad -2 - a2 = 0; \quad a2 = -2$$

$$\pi_1 = P\,\rho^{-1}\omega^{-2}D^{-2} = \frac{P}{\rho\omega^2D^2} = E_u$$

$$\pi_2 = \mu\,\rho^{b1}\omega^{b2}D^{b3}$$

$$\pi_2:\ [ML^{-1}T^{-1}][ML^{-3}]^{b1}[T^{-1}]^{b2}[L]^{b3} = [M^0L^0T^0]$$

$$M:\quad 1 + b1 = 0; \quad b1 = -1$$

$$L:\quad -1 - 3b1 + b3 = 0; \quad 2 + b3 = 0; \quad b3 = -2$$

$$T:\quad -1 - b2 = 0; \quad b2 = -1$$

$$\pi_2 = \mu\,\rho^{-1}\omega^{-1}D^{-2} = \frac{\mu}{\rho\omega D^2} = \frac{1}{R_e}$$

Therefore: $\boldsymbol{E_u = f(R_e)}$

Problem:

On worked example 12 the working fluid is air and the **rotational velocity is large enough to compressibility been significant**.

As before we are looking the **torque** on the cone; the torque on the rotating cone is related to the rotation speed, fluid properties, viscosity μ and density ρ, cone base diameter, and this time as the fluid is compressible we also need to consider the pressure $\boldsymbol{P}$ and take into account that the speed of sound is $c = \sqrt{\frac{kp}{\rho}}$, i.e. $p = \frac{1}{k}\rho c^2$.

$$\boldsymbol{T} = f(\omega, \rho, \boldsymbol{\mu}, D, \boldsymbol{P})$$

Thus we expect *n* - *k* = 6 - 3 = **3** Pi groups:

$$\pi_1 = D\,\rho^{a1}\omega^{a2}D^{a3}$$

→ **From example 12:**

$$Cm = \frac{T}{\frac{1}{2}\rho\omega^2 D^5}$$

$$\pi_2 = \mu \, \rho^{b1} \omega^{b2} D^{b3}$$

From example 12:

$$\pi_2 = \frac{\mu}{\rho \omega D^2} = \frac{1}{R_e}$$

$$\pi_3 = P \, \rho^{c1} \omega^{c2} D^{c3}$$

From the previous example:

$$\pi_3 = \frac{P}{\rho \omega^2 D^2}$$

$$\pi_3 = E_u$$

Taking into account the **speed of sound** is $c = \sqrt{\frac{kp}{\rho}}$, i.e. $\boldsymbol{p = \frac{1}{k} \rho c^2}$

$$\pi_3 = \frac{c^2}{k \omega^2 D^2} = \boldsymbol{\frac{1}{k M_a^2}};$$

compressible fluid $\boldsymbol{E_u} = \frac{1}{k M_a^2}$

→ $\boldsymbol{C_m = f(R_e, \sqrt{k}\, M_a)}$

Problem:

On a **Boundary layer** over a flat plate, the **drag force** D varies with the characteristic length of the plate L, freestream velocity U, viscosity μ, and density ρ. Find the dimensionless parameters for this problem.

Solution: The functional relationship is $\boldsymbol{D} = f(U, \boldsymbol{\mu}, \rho, L)$, with *n* = 5 variables and *k* = 3 primary dimensions (M,L,T).
Thus we expect *n* - *k* = 5 - 3 = **2** Pi groups:

$$\pi_1 = D\ \rho^{a1} U^{a2} L^{a3}$$

$$\pi_1:\ \ [MLT^{-2}][ML^{-3}]^{a1}[LT^{-1}]^{a2}[L]^{a3} = [M^0 L^0 T^0]$$

$$M:\quad 1 + a1 = 0; \qquad a1 = -1$$

$$L:\quad 1 - 3a1 + a2 + a3 = 0; \quad 4 + a2 + a3 = 0$$

$$T:\quad -2 - a2 = 0; \quad a2 = -2; \quad a3 = -2$$

$$\pi_1 = D\ \rho^{-1} U^{-2} L^{-2} = \frac{D}{\rho U^2 L^2} = \frac{D}{\rho U^2 A} = C_D/2$$

$$\pi_2 = \mu\, \rho^{b1} U^{b2} L^{b3}$$

$$\pi_2\!:\ [ML^{-1}T^{-1}][ML^{-3}]^{b1}[LT^{-1}]^{b2}[L]^{b3} = [M^0L^0T^0]$$

$$M\!:\quad 1 + b1 = 0; \qquad b1 = -1$$

$$L\!:\quad -1 - 3b1 + b2 + b3 = 0; \quad 2 + b2 + b3 = 0$$

$$T\!:\quad -1 - b2 = 0; \quad b2 = -1; \qquad a3 = -1$$

$$\pi_2 = \mu\, \rho^{-1} U^{-1} L^{-1} = \frac{\mu}{\rho U L} = \frac{1}{R_L}$$

Therefore:

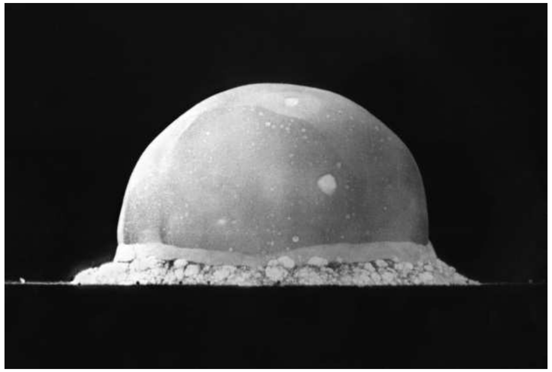
$$C_D = f(R_L)$$

Thus $C_D = f(R_L)$ just as in the boundary layer theory.

As can be observed from the above analysis, this result is also valid for any type of boundary layer.

Problem:

In an atomic explosion, a rapid release of a significant amount of energy occurs within a small region. A strong spherical shock wave develops at the point of detonation: in the early state, the pressure behind the wave front is several thousand times the initial air pressure, whose influence may neglected. Thus the radius of the shock wave front, R, at an interval of time, T,after the explosion depends on the quantity of energy release, E, the time, T, and on the initial air density, ρ.



Photograph of the fireball of the atomic explosion in New Mexico

Taylor (1941) analysed data from a series of high-speed photographs of the expansion of a fireball taken during an American nuclear test and using the π theorem concluded that the radius of the wave front satisfies the following relation:

$$R = \pi \, E^{1/5} T^{2/5} \rho^{-1/5}$$

The dimensions of the above quantities are:

$$R[L]; T[T];\ \rho[M\,L^{-3}]; E[ML^2T^{-2}]$$

Using the π theorem repeat Taylor's analysis.

$$m = n - k = 4 - 3 = 1$$

$$\pi = RT^a\rho^bE^c \to [L][T]^a[M\,L^{-3}]^b[ML^2T^{-2}]^c = [L^0T^0M^0]$$

$$L: 1 - 3b + 2c = 0$$

$$T:\ \ a - 2c = 0$$

$$M: b + c = 0$$

Solution: $a = -\frac{2}{5}, b = \frac{1}{5}, c = -\frac{1}{5}$, therefore:

$$\pi = RT^{-2/5}\rho^{1/5}E^{-1/5}, \qquad or \quad \boldsymbol{R = \pi\frac{T^{2/5}E^{1/5}}{\rho^{1/5}}}$$

with π as experimental constant.

Worked example 13

The power input to a centrifugal pump is a function of the volume flowrate, Q, impeller diameter D, rotation rate Ω and fluid properties μ and ρ.

$$P = f(Q, D, \Omega, \rho, \mu)$$

Objective of the experiment

<u>To determine P as a function Q by a *Re* similarity</u>

Re-write this as a dimensionless relationship using Ω, ρ and D as the repeating variables.

$$\frac{P}{\rho \Omega^3 D^5} = f\left(\frac{\mu}{\rho \Omega D^2}, \frac{Q}{\Omega D^3}\right)$$

Answer:

Solution to Worked Example 13

$$P = f(Q, D, \Omega, \rho, \mu)$$

- First find the dimensions of each variable

Quantity	SI unit	Dimensions
P, power	W	ML^2T^{-3}
Q, volume flowrate	m^3/s	L^3T^{-1}
ρ, density	kg/m^3	ML^{-3}
Ω, angular velocity	rad/s	T^{-1}
D, length or diameter	m	L
μ, viscosity	kg/ms	$ML^{-1}T^{-1}$

- 6 variables (n = 6)
- 3 dimensions (k = 3)
- m=6-3=3 dimensionless groups
- I choose my three variables as P, Q and μ
- The variables used to form the Π-groups are ρ, D and Ω. These cannot be used to form a non-dimensional group so are suitable.

Solution to Worked Example 13

$$[\Pi_1] = [\mathrm{P}][\rho]^{a}[\mathrm{D}]^{b}[\Omega]^{c} = [\mathrm{M}^0\mathrm{L}^0\mathrm{T}^0]$$

$$[\Pi_1] = [\mathrm{ML}^2\mathrm{T}^{-3}][\mathrm{ML}^{-3}]^{a}[\mathrm{L}]^{b}[\mathrm{T}^{-1}]^{c} = [\mathrm{M}^0\mathrm{L}^0\mathrm{T}^0]$$

- M: 1+a=0 so a = -1
- L: 2-3a+b=0
- T: -3-c=0 so c=-3
- Thus b = -5 (from the L equation)

$$\Pi_1 = \mathrm{P}\rho^{-1}\mathrm{D}^{-5}\Omega^{-3} = \frac{P}{\rho\Omega^3 D^5} = \frac{\boldsymbol{P}}{\frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{\rho}(\boldsymbol{\Omega D})^{2}\boldsymbol{D}^{2}(\boldsymbol{\Omega D})}$$

Solution to Worked Example 13

- Likewise for Π_2

$$[\Pi_2] = [\mu][\rho]^d[D]^e[\Omega]^f = [M^0L^0T^0]$$

$$[\Pi_2] = [ML^{-1}T^{-1}][ML^{-3}]^d[L]^e[T^{-1}]^f = [M^0L^0T^0]$$

- M: 1+d=0 so d=-1
- L: -1-3d+e=0
- T: -1-f=0 so f=-1
- Thus e=-2

$$\Pi_2 = \mu\rho^{-1}D^{-2}\Omega^{-1} = \frac{\mu}{\rho\Omega D^2}$$

- Again this is 1/Re, the rotating Reynolds number

Solution to Worked Example 13

- And for Π_3

$$[\Pi_3] = [Q][\rho]^g[D]^h[\Omega]^i = [M^0L^0T^0]$$

$$[\Pi_3] = [L^3T^{-1}][ML^{-3}]^g[L]^h[T^{-1}]^i = [M^0L^0T^0]$$

- M: g=0
- L: 3-3g+h=0 and as g=0 then h=-3
- T: -1-i=0 so i=-1

$$\Pi_3 = Q\rho^0D^{-3}\Omega^{-1} = \frac{Q}{\Omega D^3}$$

- Thus

$$\frac{P}{\rho\Omega^3D^5} = f\left(\frac{\mu}{\rho\Omega D^2}, \frac{Q}{\Omega D^3}\right)$$

- As will be seen in the Turbomachinery topic, these are the standard non-dimensional groups used for correlating pump power.

What about dimensionless variables?

- Sometimes there is a variable that affects the outcome but has no dimensions
  - Eg the number of blades on a wind turbine
  - Or the angle of attack on an aerofoil
- This is added in as another variable but is automatically one of the non-dimensional groups
- Looking at WE13 for example, add number of blades as a variable
  - Number of variables increases from 6 to 7
  - Number of dimensions remains 3
  - Number of groups increases from 3 to 4
  - Additional group is N
  - Functional relationship becomes:

$$\frac{P}{\rho\Omega^3 D^5} = f\left(\frac{\mu}{\rho\Omega D^2}, \frac{\mathrm{Q}}{\Omega \mathrm{D}^3}, N\right)$$

Problem:

A liquid of density ρ and viscosity μ flows in a channel of width H with a characteristic velocity U, one small side of the upper wall is heated at a temperature T_w with respect to the inlet flow temperature T_0 while the rest of the walls are thermally isolated. Knowing that thermal conductivity of the liquid k has dimension $[M\ L\ T^{-3}\Theta^{-1}]$ and its specific heat c_p has dimension $[L^2\ T^{-2}\Theta^{-1}]$ find a dimensionless relation for the wall **heat flux** q of dimension $[M\ T^{-3}]$.

$$\boldsymbol{q} = f(\boldsymbol{k}, c_p, \rho, \boldsymbol{\mu}, H, U, (\boldsymbol{T_w} - \boldsymbol{T_0}))$$

Units (dimensions) → $M\ L\ T\ \Theta$,

$$m = n - k = 8 - 4 = 4$$

Quantity of interest ($\boldsymbol{q}$) and free parameters ($\boldsymbol{k}$, $\boldsymbol{\mu}$, $(\boldsymbol{T_w} - \boldsymbol{T_0})$) in the characteristic dimensionless numbers **(Péclet, Reynolds and dimensionless heat radiation).**

We are looking for a dimensionless heat flux in terms of the imposed dimensionless increase of temperature, and according to the governing equations we expect similarity with respect to the Reynolds and Peclet numbers.

$$\pi_1 = q\, \rho^{a1} U^{a2} H^{a3} c_p^{a4}$$

$$\pi_1: \; [M\, T^{-3}][ML^{-3}]^{a1}[LT^{-1}]^{a2}[L]^{a3}[L^2\, T^{-2}\Theta^{-1}]^{a4} = [M^0 L^0 T^0]$$

$$M: \quad 1 + a1 = 0; \qquad a1 = -1$$

$$L: \quad 0 - 3a1 + a2 + a3 + 2a4 = 0; \quad 3 + a2 + a3 + 2a4 = 0; \;\; a3 = 0$$

$$T: \;\; -3 - a2 - 2a4 = 0; \;\; a2 = -3 \;\uparrow$$

$$\Theta: 0 - a4 = 0; \;\; a4 = 0 \;\uparrow$$

$$\pi_1 = q\, \rho^{-1} U^{-3} = \frac{q}{\rho U^3}$$

$$\pi_2 = \; k\, \rho^{b1} U^{b2} H^{b3} c_p^{b4}$$

$$\pi_2: [M\, L\, T^{-3}\Theta^{-1}]\, [ML^{-3}]^{b1}[LT^{-1}]^{b2}[L]^{b3}[L^2\, T^{-2}\Theta^{-1}]^{b4} = [M^0 L^0 T^0]$$

$$M: \quad 1 + b1 = 0; \qquad b1 = -1$$

$$L: \quad 1 - 3b1 + b2 + b3 + 2b4 = 0; \quad 4 + b2 + b3 + 2b4 = 0; \;\; b3 = -1$$

$$T: \;\; -3 - b2 - 2b4 = 0; \quad b2 = -1 \;\; \uparrow$$

$$\Theta: -1 - b4 = 0; \;\; b4 = -1 \;\uparrow$$

$$\pi_2 = k\, \rho^{-1} U^{-1} H^{-1} c_p^{-1} = \frac{k}{\rho U H c_p}; \;\; D = \frac{k}{\rho c_p} \;\rightarrow \pi_2 = \frac{D}{UH} = \frac{1}{P_e}$$

$$\pi_3 = \mu\, \rho^{c1} U^{c2} H^{c3} c_p^{c4}$$

$$\pi_3\text{: } [M\, L^{-1}\, T^{-1}]\, [ML^{-3}]^{c1} [LT^{-1}]^{c2} [L]^{c3} [L^2\, T^{-2} \Theta^{-1}]^{c4} = [M^0 L^0 T^0]$$

$$M\text{:} \quad 1 + c1 = 0; \quad c1 = -1$$

$$L\text{:} \quad -1 - 3c1 + c2 + c3 + 2c4 = 0; \quad 2 + c2 + c3 + 2c4 = 0; \quad c3 = -1$$

$$T\text{:} \quad -1 - c2 - 2c4 = 0; \quad c2 = -1 \ \uparrow$$

$$\Theta\text{: } 0 - c4 = 0; \quad c4 = 0 \ \uparrow$$

$$\pi_3 = \mu\, \rho^{-1} U^{-1} H^{-1} = \frac{\mu}{\rho U H}; \quad \pi_3 = \frac{1}{R_e}$$

$$\pi_4 = (T_w - T_0)\, \rho^{d1} U^{d2} H^{d3} c_p^{d4}$$

$$\pi_2\text{: } [\Theta]\, [ML^{-3}]^{d1} [LT^{-1}]^{d2} [L]^{d3} [L^2\, T^{-2} \Theta^{-1}]^{d4} = [M^0 L^0 T^0]$$

$$M\text{:} \quad 0 + d1 = 0; \quad d1 = 0$$

$$L\text{:} \quad 0 - 3d1 + d2 + d3 + 2d4 = 0; \quad d2 + d3 + 2d4 = 0; \quad d3 = 0$$

$$T\text{: } -d2 - 2d4 = 0; \quad d2 = -2 \ \uparrow$$

$$\Theta\text{: } 1 - d4 = 0; \quad d4 = 1 \ \uparrow$$

$$\pi_4 = (T_w - T_0)\, U^{-2} c_p = \frac{(T_w - T_0)\, c_p}{U^2}$$

Therefore: $\frac{q}{\rho U^3} = f\left(R_e, P_e, \frac{(T_w - T_0)\, c_p}{U^2}\right)$

Problem:
When tested in water at 20°C flowing at 2 m/s, an 8-cm-diameter sphere has a measured drag of 5 N ($kg\ m\ s^{-2}$). What will be the velocity and drag force on a 1.5-m-diameter weather balloon moored in sea-level standard air under dynamically similar conditions?

Solution: For water at 20°C take $\rho = 998\ \frac{kg}{m^3}$ and $\mu = 0.001\ \frac{kg}{m\ s}$. For sea-level standard air take $\rho = 1.2255\ \frac{kg}{m^3}$ and $\mu = 1.78\ 10^{-5}\ \frac{kg}{m\ s}$.

The balloon velocity follows from *dynamic similarity*, which requires identical Reynolds numbers:

$$R_e\Big|_m = R_e\Big|_p$$

$$R_e\Big|_m = \frac{\rho U D}{\mu}\Bigg|_m = \frac{998x2x0.08}{0.001} = 1.6\ 10^5$$

$$R_e\Big|_p = \frac{\rho U D}{\mu}\Bigg|_p = \frac{1.2255\ x\ U_p\ x\ 1.5}{1.78\ 10^{-5}} = 1.6\ 10^5$$

or $U_p = \boldsymbol{U_{balloon}}$ =1.55 m/s.

Then the two spheres will have identical drag coefficients:

$$C_D = f(R_e)$$

$$\frac{D}{\rho U^2 A} = 2C_D$$

$$C_D\Big|_m = C_D\Big|_p$$

$$C_D\Big|_m = \frac{D}{\frac{1}{2}\rho U^2 D^2}\Bigg|_m = \frac{5}{\frac{1}{2}998\ x\ 2^2\ x\ 0.08^2} = 0.396$$

$$D\Big|_p = \frac{1}{2}\rho U^2 D^2\Big|_p\ C_D\Bigg|_m = \frac{1}{2}\ 1.2255\ x\ 1.55^2\ x\ 1.5^2\ x\ 0.396 = 1.3\ N$$

Therefore $D_{balloon}$ **= 1.3 N**.

Worked example 14

A prototype boat is to be tested at a model scale of 50:1. This is an incompressible free surface flow and to ensure dynamic similarity both Reynolds number and Froude number must be equal. What kinematic viscosity must the model working fluid have if the prototype working fluid is water (ν=10^{-6} m^2/s) ?

Answer: $\nu_m = 2.83 \times 10^{-9}\ m^2/s$

Solution to Worked Example 14

- We want $Fr_m = Fr_p$ where $Fr = \frac{U^2}{gL}$
- So $\frac{U_m^2}{gL_m} = \frac{U_p^2}{gL_p}$
- Geometric similarity gives us $\frac{L_p}{L_m} = 50$
- Dynamic similarity in this case tells us that we must have $\frac{U_p^2}{U_m^2} = \frac{L_p}{L_m} = 50$
- So $\boldsymbol{\frac{U_p}{U_m} = \sqrt{50}}$

Solution to Worked Example 14

- Now thinking about Re, $Re = \frac{\rho UL}{\mu} = \frac{UL}{\nu}$
- We want: $Re_p = Re_m$
- Ie $\frac{U_m L_m}{\nu_m} = \frac{U_p L_p}{\nu_p}$
- So we need $\frac{U_m L_m}{\nu_m} = \frac{U_p L_p}{\nu_p}$
- Giving $\frac{\nu_m}{\nu_p} = \frac{U_m}{U_p}\frac{L_m}{L_p} = \frac{1}{50\sqrt{50}} = 0.00283$
- So $\nu_m = 2.83 \times 10^{-9}\ m^2/s$ = 0.00283 cSt
- The closest liquid is mercury with $\nu = 1.18 \times 10^{-7}$ m^2/s (0.118 cSt).
- However mercury is not a suitable experimental fluid for this problem and we need to find another scale for the model.

1 cSt = 10^{-6} m^2/s

Fluid	Temperature (°C)	Kinematic viscosity (cSt)
Distilled water	20	1.004
Olive oil	37.8	43.2
mercury	21.1	0.118
kerosene	20	2.71
Freon 11	21.1	0.21
Acetone	20	0.41